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ZEROS OF ABELIAN INTEGRALS FOR THE REVERSIBLE CODIMENSION FOUR QUADRATIC CENTERS $Q_3^R \cap Q_4$*

BY

YULIN ZHAO**

Department of Mathematics, Sun Yat-sen University
Guangzhou, 510275, People's Republic of China
e-mail: mcszyl@zsu.edu.cn

ABSTRACT

We study the number of zeros of Abelian integrals for the reversible codimension four quadratic centers $Q_3^R \cap Q_4$, when we perturb such systems inside the class of all polynomial systems of degree n.

1. Introduction

The paper is concerned with the Abelian integrals for the perturbations of the planar polynomial system

$$(1.1)_\epsilon \qquad \begin{cases} \dot{x} = H_y/M + \epsilon f(x,y), \\ \dot{y} = -H_x/M + \epsilon g(x,y), \end{cases}$$

where ϵ is a small parameter, $H(x,y)$ is a first integral of system $(1.1)_0$ and $M(x,y)$ is the integrating factor, $H_y/M, H_x/M, f(x,y)$ and $g(x,y)$ are polynomials of x and y, $max\{\deg f(x,y), \deg g(x,y)\} = n$. The system $(1.1)_0$ is called an **integrable system** (**Hamiltonian system**, if $M(x,y) \equiv 1$). We assume

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that system $(1.1)_0$ has at least one center. The Abelian integral for $(1.1)_\epsilon$ is defined as

$$(1.2) \qquad I(h)=\oint_{\Gamma_h} -M(x,y)f(x,y)dy+M(x,y)g(x,y)dx, \quad h\in\Sigma.$$

Here Γ_h is the compact component of the curve $H(x,y)=h$ (i.e., the period annulus of system $(1.1)_0$); Σ is the maximal interval of existence of Γ_h. Finding the upper bound for the number of zeros of $I(h)$ is called the weakened Hilbert 16th problem, posed by Arnold [A]. It is well known that, if $I(h)\not\equiv 0$, then the total number of isolated zeros of $I(h)$ (taking into account their multiplicities) is an upper bound for the number of limit cycles of $(1.1)_\epsilon$, which tend to some period annulus Γ_h.

Up to now most of the results on the weakened Hilbert 16th problem concerned the Hamiltonian cases; see [G1, G2, HI, I3, P1, P2, RZh, ZLL, ZZ, IyY1, IyY2, Y, NY] and references therein. For non-Hamiltonian integrable systems, since $M(x,y)$ is not a constant, the functions $M(x,y)f(x,y), M(x,y)g(x,y)$ and $H(x,y)$ are in general not polynomials. The study of Abelian integrals in these case is much more difficult than the Hamiltonian cases. Only a few papers dealt with integrable cases; see [I1, I2, Zh, ZLLZ] etc.

In this paper, we discuss the Abelian integral for a quadratic integrable system. The quadratic centers are divided into several types. The most simple classification can be found in [Zh]. Taking a complex $z=x+iy$ and using the terminology from [Zh], the list of quadratic centers at $z=0$ looks therefore as follows:

$\dot{z}=-iz-z^2+2|z|^2+(b+ic)\bar{z}^2$, Hamiltonian (Q_3^H),

$\dot{z}=-iz+az^2+2|z|^2+b\bar{z}^2$, reversible (Q_3^R),

$\dot{z}=-iz+4z^2+2|z|^2+(b+ic)\bar{z}^2, |b+ic|=2$, codimension four (Q_4),

$\dot{z}=-iz+z^2+(b+ic)\bar{z}^2$, generalized Lotka-Volterra (Q_3^{LV}),

$\dot{z}=-iz+\bar{z}^2$, Hamiltonian triangle.

Important work was finished in the paper [I2] by Iliev, who studied the bifurcation of limit cycles in the general quadratic perturbation of a quadratic integrable system and gave the corresponding bifurcation functions, which are Abelian integral $I(h)$ or higher order Melnikov functions. For the cases of the Hamiltonian triangle and Q_3^H, the number of zeros of $I(h)$ has been estimated in [G1] and [HI], respectively.

The present paper deals with the quadratic case in the situation where unperturbed vector field $(1.1)_0$ belongs to the intersection of two components of a center manifold, namely the reversible Q_3^R and the codimension four Q_4. The intersection $Q_3^R \cap Q_4$ consists of two single systems $Q_4^{\pm}$ (see [I1])

$$\dot{z} = -iz + 4z^2 + 2|z|^2 \pm 2\bar{z}^2. \tag{1.3}$$

In the papers [I1] and [I2], the author proved that the cyclicity of the period annulus of system $Q_4^{\pm}$ is at most three under quadratic perturbations. The purpose of this paper is to find an upper bound of the number of zeros of Abelian integral (1.2) for $Q_4^{\pm} = Q_3^R \cap Q_4$ when we perturb such a system inside the class of all polynomial systems of degree n. Our main result is the following

THEOREM 1.1: *The number of isolated zeros of Abelian integral $I(h)$ in Σ does not exceed $5[(n-1)/2]-1, n \geq 3$, for system Q_4^+ and $21n-12, n \geq 1$, for Q_4^-, respectively.*

For $n = 1, 2$, $I(h)$ has at most 4 zeros in Σ for system Q_4^+.

Remark 1.2: In the paper [ZLLZ], we give a linear estimate of the number of zeros of Abelian integrals for quadratic centers having almost all their orbits formed by cubic curves. However, Q_4^- is not contained in any cases which are studied in [ZLLZ], although it can be reduced to a cubic integrable vector field whose orbits are almost all cubic curves; see section 2 for the details. This is because the original orbits of Q_4^- are almost all algebraic curves of degree 6.

2. The expression for Abelian integrals

In this section, we are going to express the Abelian integral $I(h)$ as a linear combination of several basic integrals. To do this, we begin this section with the following lemma:

LEMMA 2.1: (i) *Corresponding to system Q_4^-, the perturbed system $(1.1)_\epsilon$ can be reduced to the following normal form,*

$$(2.1)_\epsilon \qquad \begin{cases} \dot{x} = xy + \epsilon x^{-2} f(x^3, y), \\ \dot{y} = y^2 - x^3/3 + 1/3 + \epsilon g(x^3, y). \end{cases}$$

A first integral of system $(2.1)_0$ is

$$H(x,y) = x^{-2}\Big(\frac{1}{2}y^2 + \frac{1}{3}x^3 + \frac{1}{6}\Big) = h \tag{2.2}$$

with integrating factor $M(x,y) = x^{-3}$. In these coordinates, the ovals $\Gamma_h \subseteq \{H = h\}$ are defined for $h \in \Sigma = (1/2, +\infty)$ and the critical level $H = 1/2$ corresponds to the center $(1,0)$.

(ii) *Corresponding to system Q_4^+, the perturbed system $(1.1)_\epsilon$ can be reduced to*

$$(2.3)_\epsilon \qquad \begin{cases} \dot{x} = 2xy + \epsilon f(x,y), \\ \dot{y} = 2(x - x^2 + 2y^2) + \epsilon g(x,y). \end{cases}$$

A first integral of $(2.3)_0$ is

$$(2.4) \qquad H(x,y) = x^{-4}\Big(y^2 - x^2 + \frac{2}{3}x\Big) = h$$

with integrating factor $M(x,y) = x^{-5}$. The period annulus Γ_h is defined in the interval $\Sigma = (-1/3, 0)$ and the critical level $H = -1/3$ corresponds to the center $(1,0)$.

In the above system $(2.1)_\epsilon$ and $(2.3)_\epsilon$, $f(x,y)$ and $g(x,y)$ are polynomials of x, y with $max\{\deg f(x,y), \deg g(x,y)\} = n$.

Proof: (i) Taking a real coordinate (x,y), the system Q_4^- has a first integral $H = X^{-2/3}(y^2/2 + X/48 + 1/96)$ with integrating factor $M = X^{-5/3}$, where $X = 1 + 12x$; see appendix of [I1]. Performing a suitable scaling of y, H and h, we can assume $H = x^{-2/3}(y^2/2 + x/3 + 1/6) = h$ with $M = x^{-5/3}$. Changing x into x^3, we get $(2.1)_\epsilon$.

(ii) Using the same arguments as above.

Now we introduce some notations. Throughout this paper, we define, for $h \in \Sigma$ and any i, j, $i = \ldots, -1, 0, 1, \ldots$, $j = 0, 1, 2, \ldots$,

$$(2.5) \qquad I_{i,j}(h) = \oint_{\Gamma_h} M(x,y) x^i y^j dx, \; J_i(h) = I_{i,1}(h),$$

where Γ_h, Σ and $M(x,y)$ are defined in Lemma 2.1 (i) and Lemma 2.1 (ii), respectively. Without loss of generality, suppose that Γ_h has negative (clockwise) orientation. Obviously, $I_{i,2k}(h) \equiv 0, k = 0, 1, 2, \ldots$ for both cases.

To be more concrete, in the following we only consider the case Q_4^- (i.e., system $(2.1)_\epsilon$) in Lemma 2.2–2.3 and Proposition 2.4.

LEMMA 2.2: *The Abelian integral $I(h)$, related to $(2.1)_\epsilon$, can be expressed in the form*

$$(2.6) \qquad I(h) = \sum_{i=-1}^{n-1} c_i J_{3i} + \eta I_{-3,5},$$

where η and $c_i, i = -1, 0, 1, \ldots$ are real constants, $n \geq 4$.

Proof: By partial integration, we get

$$\oint_{\Gamma_h} Mx^{-2}x^{3i}y^j dy = \frac{1}{j+1}\oint_{\Gamma_h} x^{3i-5}dy^{j+1} = -\frac{3i-5}{j+1}I_{3(i-1),j+1}. \tag{2.7}$$

Therefore, we only consider $I_{3i,j}(h), i \geq -1$.

It follows from (2.2) that

$$x^{-2}y\frac{\partial y}{\partial x} - x^{-3}y^2 + \frac{1}{3} - \frac{1}{3}x^{-3} = 0. \tag{2.8}$$

Multiplying (2.8) by $x^i y^{j-2}dx$ and integrating over Γ_h, we get

$$\oint_{\Gamma_h} x^{i-2}y^{j-1}dy - I_{i,j} + \frac{1}{3}(I_{i+3,j-2} - I_{i,j-2}) = 0. \tag{2.9}$$

By partial integrations, we get from (2.9) that

$$\frac{i+j-2}{j}I_{i,j} = \frac{1}{3}(I_{i+3,j-2} - I_{i,j-2}). \tag{2.10}$$

If $i \geq 0, j \geq 3$ and j is odd, then $i+j-2 > 0$. By induction for j, we obtain from (2.10) that $I_{i,j}$, $i \geq 0, j \geq 3$ and j is odd, can be expressed as

$$I_{i,j} = \sum_{k=0}^{(j-1)/2} \bar{c}_i J_{i+3k}, \tag{2.11}$$

where $\bar{c}_i$ denotes real constants. It follows from (2.10) that $I_{-3,j}$, j is odd, $j \geq 3$, can be expressed in the form

$$I_{-3,j} = \sum_{k=5}^{j-2} b_k I_{0,k} + \bar{b}I_{-3,5}, \quad j \geq 7, \tag{2.12}$$

where b_k and $\bar{b}$ are real constants, k is odd. Introducing $(i,j) = (-3,5)$ into (2.10), we have

$$I_{-3,3} = I_{0,3}. \tag{2.13}$$

The expression (2.6) follows from (2.11), (2.12) and (2.13).

LEMMA 2.3: *$J_i(h), i \geq 4$, related to (2.2), can be expressed in the form*

$$J_i(h) = \alpha_{i,0}(h)J_0 + \beta_{i,1}(h)J_1 + \gamma_{i,2}(h)J_2, \tag{2.14}$$

where $\alpha_{i,0}(h), \beta_{i,1}(h)$ and $\gamma_{i,2}(h)$ are polynomials of h with $\deg \alpha_{i,0}(h) \leq i - 3, \deg \beta_{i,1}(h) \leq i-4, \deg \gamma_{i,2}(h) \leq i-2$. For $i = -1, -3, 3$,

$$J_{-1} = J_2, \quad J_{-3} = \frac{1}{5}(-J_0 + 12hJ_2), \quad J_3 = \frac{1}{5}(2J_0 + 6hJ_2). \tag{2.15}$$

Proof: Rewrite (2.2) in the form

$$\frac{1}{2}y^2 + \frac{1}{3}x^3 + \frac{1}{6} = hx^2,$$

which yields

$$I_{i,j} = 3hI_{i-1,j} - \frac{3}{2}I_{i-3,j+2} - \frac{1}{2}I_{i-3,j}. \tag{2.16}$$

It follows from (2.10) that

$$3(i+j-3)I_{i-3,j+2} = (j+2)(I_{i,j} - I_{i-3,j}). \tag{2.17}$$

Introducing $(i,j) = (2,1)$ into (2.17), one obtains $J_{-1} = J_2$. Eliminating $I_{i-3,j+2}$ from (2.16) and (2.17), we have

$$(2i+3j-4)I_{i,j} = 6h(i+j-3)I_{i-1,j} + (-i+5)I_{i-3,j}, \tag{2.18}$$

which implies

$$J_i = \frac{1}{2i-1}[6h(i-2)J_{i-1} + (-i+5)J_{i-3}]. \tag{2.19}$$

The results of this lemma follow from (2.19) by induction for i.

PROPOSITION 2.4: *If $n \geq 3$, then the Abelian integral $I(h)$, related to $(2.1)_\epsilon$, can be expressed in the form*

$$I(h) = \alpha(h)J_0 + \beta(h)J_1 + \gamma(h)J_2, \tag{2.20}$$

where $\alpha(h)$, $\beta(h)$ and $\gamma(h)$ are polynomials of h with $\deg\alpha(h) \leq 3n-6$, $\deg\beta(h) \leq 3n-7$ and $\deg\gamma(h) \leq 3n-5$.

For $n = 0, 1, 2$, $\deg\alpha(h) = 0$, $\beta(h) \equiv 0$ and $\deg\gamma(h) \leq 1$.

Proof: It follows from (2.10), (2.13), (2.15) and (2.16) that

$$I_{0,3} = J_3 - J_0, \quad I_{-3,5} = 2hI_{-1,3} - I_{0,3}, \quad I_{-1,3} = 2hJ_1 - J_2. \tag{2.21}$$

By (2.21) and (2.15), we have

$$I_{-3,5} = \frac{3}{5}J_0 + 4h^2J_1 - \frac{16}{5}hJ_2. \tag{2.22}$$

The proposition follows from Lemma 2.2, Lemma 2.3 and (2.21), (2.22).

PROPOSITION 2.5: *If $n \geq 7$, then the Abelian integral $I(h)$, related to $(2.3)_\epsilon$, is expressed in the form*

$$I(h) = \frac{1}{h^{[(n-3)/2]}} J(h), \quad J(h) = \alpha(h)J_0 + \beta(h)J_1 + \gamma(h)J_2, \tag{2.23}$$

where $\alpha(h), \beta(h)$ and $\gamma(h)$ are polynomials of h with

$$max\{\deg\alpha(h), \deg\beta(h), \deg\gamma(h)\} \leq [(n-3)/2];$$

$[s]$ denotes the entire part of s.

If $n = 1, 2, 3$, then $I(h) = J(h)$ with $\deg\alpha(h) = \deg\beta(h) = \deg\gamma(h) = 0$; if $n = 4, 5, 6$, then $I(h) = J(h)/h$ with $\deg\alpha(h) = \deg\beta(h) = \deg\gamma(h) = 0$.

Proof: Using the same arguments as in the proof of Proposition 2.4, we can get (2.23). Hence, we only sketch here the outline of the proof.

At first, we prove that $I(h)$ can be denoted in the form $I(h) = \sum_{i=-1}^{n-1} c_i J_i$, where $c_i, i = -1, 0, 1, \ldots$, is a real constant. Then we obtain

$$3h(i-2)J_i = -3(i-5)J_{i-2} + (2i-13)J_{i-3},$$

which means $J_i(h)$ can be expressed in the form

$$J_{-1} = J_0, J_i(h) = \frac{1}{h^{[(i-2)/2]}}(\alpha_{i,0}(h)J_0 + \beta_{i,1}(h)J_1 + \gamma_{i,2}(h)J_2), i \geq 4.$$

Here $\alpha_{i,0}(h), \beta_{i,1}(h), \gamma_{i,2}(h)$ are polynomials of h. If $4 \leq i \leq 7$, then $\deg\alpha_{i,0}(h) = \deg\beta_{i,1}(h) = \deg\gamma_{i,2}(h) = 0$; if $i \geq 8$, then $\deg\alpha_{i,0}(h) \leq [(i-8)/6] + \mathcal{F}(i-6)$, $\deg\beta_{i,1}(h) \leq [(i-6)/6] + \mathcal{F}(i-4)$, $\deg\gamma_{i,1}(h) \leq [(i-4)/6] + \mathcal{F}(i-2)$, where

$$\mathcal{F}(i) = \begin{cases} 1, & \text{if } i = 6k, \\ 0, & \text{if } i \neq 6k, k = 0, 1, 2, \ldots. \end{cases}$$

For $i = 3$, we have

$$J_3 = \frac{1}{3h}(-7J_0 + 6J_1).$$

Using the above results, we get (2.23).

3. The Picard–Fuchs equation and relevant results

In this section, we derive the Picard–Fuchs equation satisfied by J_0, J_1 and J_2. This is crucial for our analysis.

LEMMA 3.1: *The Abelian integrals $J_0(h), J_1(h), J_2(h)$, related to (2.2), satisfy the following Picard–Fuchs equation:*

$$\begin{pmatrix} J_0 \\ J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} h & -1/2 & 0 \\ 0 & 2h & -1 \\ -1/3 & 0 & 2h/3 \end{pmatrix} \begin{pmatrix} J_0' \\ J_1' \\ J_2' \end{pmatrix}. \tag{3.1}$$

Proof: It follows from (2.2) that

$$\frac{\partial y}{\partial h} = \frac{x^2}{y}, \tag{3.2}$$

which implies

$$J_i'(h) = \oint_{\Gamma_h} x^{i-3} \frac{\partial y}{\partial h} dx = \oint_{\Gamma_h} \frac{x^{i-1}}{y} dx. \tag{3.3}$$

Using (2.2) again, we get

$$\begin{aligned} J_i(h) = & \oint_{\Gamma_h} \frac{x^{i-3} y^2}{y} dx = \oint_{\Gamma_h} \frac{x^{i-3}(2hx^2 - 2x^3/3 - 1/3)}{y} dx \\ = & 2hJ_i' - \frac{2}{3} J_{i+1}' - \frac{1}{3} J_{i-2}'. \end{aligned} \tag{3.4}$$

Substituting $i = 0$ into (3.4), we get

$$J_0 = 2hJ_0' - \frac{2}{3} J_1' - \frac{1}{3} J_{-2}'. \tag{3.5}$$

The equality (2.19) yields

$$J_{-2} = \frac{3}{2} h J_0 + \frac{1}{4} J_1. \tag{3.6}$$

Inserting (3.6) into (3.5), one gets the first equation of (3.1). By the same arguments, the second and third equations follow.

COROLLARY 3.2: *The Abelian integrals J_0, J_1, J_2, related to (2.2), satisfy the following equation:*

$$(8h^3 - 1) \begin{pmatrix} J_0'' \\ J_1'' \\ J_2'' \end{pmatrix} = \begin{pmatrix} -2h & 1 \\ -4h^2 & 2h \\ -1 & 4h^2 \end{pmatrix} \begin{pmatrix} J_1' \\ J_2' \end{pmatrix}. \tag{3.7}$$

Proof: Differentiating both sides of (3.1), we get

$$\begin{pmatrix} 0 \\ -J_1' \\ \frac{1}{3} J_2' \end{pmatrix} = \begin{pmatrix} h & -1/2 & 0 \\ 0 & 2h & -1 \\ -1/3 & 0 & 2h/3 \end{pmatrix} \begin{pmatrix} J_0'' \\ J_1'' \\ J_2'' \end{pmatrix},$$

which implies (3.7).

LEMMA 3.3: *The following Picard–Fuchs equation is satisfied by the Abelian integrals J_0, J_1, J_2, related to (2.4):*

$$6h(3h+1)\begin{pmatrix} J_0' \\ J_1' \\ J_2' \end{pmatrix} = \begin{pmatrix} 21h & -3h & 0 \\ -7 & 3(5h+2) & 0 \\ -7 & 1 & 3(3h+1) \end{pmatrix} \begin{pmatrix} J_0 \\ J_1 \\ J_2 \end{pmatrix}. \tag{3.8}$$

Proof: Use the same arguments as in the proof of Lemma 3.1.

For the integrable system $(2.1)_0$, the period annulus Γ_h is in the right half-plane, which means

$$J_i'(h) = 2\int_{x_1(h)}^{x_2(h)} \frac{x^{i-1}}{\sqrt{2hx^2 - 2x^3/3 - 1/3}}dx > 0,$$

where $(x_1(h), 0)$ and $(x_2(h), 0)$ are intersection points of Γ_h and the x-axis. Using the same arguments, we know that $J_i(h)$, related to $(2.3)_0$, satisfies $J_i(h) > 0$. Hence, we can define, related to $(2.1)_\epsilon$ and $(2.3)_\epsilon$ respectively,

$$\omega(h) = \frac{J_2'(h)}{J_1'(h)}, \quad v(h) = \frac{J_1(h)}{J_0(h)}. \tag{3.9}$$

By (3.7) and (3.8), we get

COROLLARY 3.4: (i) *The ratio $\omega(h) = J_2'/J_1'$, related to $(2.1)_\epsilon$, satisfies the following Riccati equation:*

$$(8h^3 - 1)\omega' = -2h\omega^2 + 8h^2\omega - 1. \tag{3.10}$$

(ii) *The ratio $v(h) = J_1/J_0$, related to $(2.3)_\epsilon$, satisfies the following equation:*

$$6h(3h+1)v' = 3hv^2 - 6(h-1)v - 7. \tag{3.11}$$

4. Estimation for Q_4^-

In this section, we investigate the number of zeros of $I(h)$ for Q_4^-. To do this, we reduce the initial problem to counting the number of isolated zeros of a certain integral which is expressed as a linear combination of only two basic integrals, J_1' and J_2'.

In this and the next section, $\alpha_i(h), \beta_i(h), \gamma_i(h)$, $i = 0, 1, 2, \ldots$, denote polynomials of h and $\#\phi(h)$ denotes the number of zeros of $\phi(h)$.

It follows from (2.20) and (3.1) that $I(h)$ and $I'(h)$ can be expressed in the form

$$\begin{aligned} I(h) &= \alpha_0(h)J_0' + \beta_0(h)J_1' + \gamma_0(h)J_2', \\ I'(h) &= \alpha_1(h)J_0' + \beta_1(h)J_1' + \gamma_1(h)J_2', \end{aligned} \tag{4.1}$$

where $\deg\alpha_0(h) \le 3n-5$, $\deg\beta_0(h) \le 3n-6$, $\deg\gamma_0(h) \le 3n-4$, $\deg\alpha_1(h) \le 3n-6$, $\deg\beta_1(h) \le 3n-7$, $\deg\gamma_1(h) \le 3n-5$. Eliminating J_0' from the above two equations, we have

$$\alpha_0(h)I'(h) = \alpha_1(h)I(h) + M(h), \tag{4.2}$$

where $M(h)$ has the form

$$M(h) = \beta_2(h)J_1' + \gamma_2(h)J_2' \tag{4.3}$$

with $\deg\beta_2(h) \le 6n-12$ and $\deg\gamma_2(h) \le 6n-10$.

In what follows we study the relation of $\#I(h)$ and $\#M(h)$. Suppose h_1 and h_2 are two consecutive simple zeros of $I(h)$; then $I'(h_1)I'(h_2) < 0$. By (4.2), we know that

$$\alpha_0(h_i)I'(h_i) = M(h_i), \quad i = 1, 2.$$

Hence, either $\alpha_0(h)$ has at least one zero in (h_1, h_2) or $M(h_1)M(h_2) < 0$, which implies that there exists $h^* \in (h_1, h_2)$ such that $\alpha_0(h^*) = 0$ or $M(h^*) = 0$. On the other hand, if $I(\bar{h}) = I'(\bar{h}) = \cdots = I^{(k)}(\bar{h}) = 0, k \ge 1$, then $M(\bar{h}) = 0$. Therefore, between any two consecutive zeros (taking into account their multiplicities) of $I(h)$, there must exists at least one zero of $\alpha_0(h)$ or $M(h)$. This means

$$\#I(h) \le \#\alpha_0(h) + \#M(h) + 1. \tag{4.4}$$

Finally, we only need to consider $\#M(h)$. Denote

$$\psi(h) = \frac{M(h)}{J_1'(h)} = \beta_2(h) + \gamma_2(h)\omega. \tag{4.5}$$

Obviously, $\#\psi(h) = \#M(h)$. It follows from (3.10) that $\psi(h)$ satisfies the following Riccati equation:

$$(8h^3 - 1)\gamma_2(h)\psi' = -2h\psi^2 + R_1(h)\psi + R_2(h), \tag{4.6}$$

where $R_1(h)$ and $R_2(h)$ are polynomials of h, $\deg R_2(h) \le 12n-20$. By the same arguments as in the proof of (4.4), we have

$$\#\psi(h) \le \#((8h^3 - 1)\gamma_2(h)) + \#R_2(h) + 1. \tag{4.7}$$

Since $I(h)$ is defined in $\Sigma = (1/2, +\infty)$, we conclude that $\#(8h^3 - 1) = 0$ in Σ. The inequalities (4.4) and (4.7) imply that

$$\begin{aligned} \#I(h) &\le \#\alpha_0(h) + \#\gamma_2(h) + \#R_2(h) + 2 \\ &\le \deg\alpha_0(h) + \deg\gamma_2(h) + \deg R_2(h) + 2 \\ &\le 21n - 33, \end{aligned}$$

where $n \geq 3$. Using the same arguments, we get $\#I(h) = 0$ for $n = 0$ and $\#I(h) \leq 9$ for $n = 1, 2$. The proof for Q_4^- is finished.

5. Estimation for Q_4^+

For the case Q_4^+, we will get a better upper bound of $\#I(h)$ (i.e. $\#J(h)$) by using the argument principle. As in the last section, we are going to reduce the initial problem to counting the number of zeros of a certain Abelian integral which is a combination of only two basic integrals, J_0 and J_1. From (2.23) and (3.8) we have

$$6h(3h+1)J'(h) = \alpha_1(h)J_0 + \beta_1(h)J_1 + \gamma_1(h)J_2, \tag{5.1}$$

where $\max\{\deg \alpha_1(h), \deg \beta_1(h), \deg \gamma_1(h)\} \leq [(n-1)/2]$. Eliminating J_2 from (5.1) and (2.23), one gets

$$6h(3h+1)\gamma(h)J' = \gamma_1(h)J + G(h). \tag{5.2}$$

Here $G(h)$ has the form

$$G(h) = \alpha_2(h)J_0 + \beta_2(h)J_1 \tag{5.3}$$

with $\max\{\deg \alpha_2(h), \deg \beta_2(h)\} = 2[(n-1)/2] - 1$. By the same arguments as in section 4, we have

$$\#J(h) \leq \#G(h) + \#\gamma(h) + 1. \tag{5.4}$$

From now on we begin to estimate $\#G(h)$ by the argument principle. Let $\widetilde{J}_i(h)$, $i = 0, 1$, be the analytic continuation of $J_i(h)$ from Σ to complex domain $\mathbf{C}$. This means that $\widetilde{J}_0$ and $\widetilde{J}_1$ satisfy (3.8) and $\widetilde{J}_i(h)|_{h\in\Sigma} = J_i(h)$.

LEMMA 5.1: (i) *$\widetilde{J}_i(h)$, $i = 0, 1$, is analytic at $h = -1/3$ and $\widetilde{J}_1(h)/\widetilde{J}_0(h) \to 1$ as $h \to -1/3$.*

(ii) *$\widetilde{J}_i(h)$, $i = 0, 1$, has the following expansion near $h = 0$:*

$$\begin{pmatrix} \widetilde{J}_0 \\ \widetilde{J}_1 \end{pmatrix} = c_1^0 \begin{pmatrix} 1 + \frac{35}{12}h + \frac{35}{288}h^2\ln(-h) + \cdots \\ \frac{7}{6} - \frac{35}{72}h\ln(-h) + \frac{245}{72}h + \frac{105}{576}h^2 ln(-h) + \cdots \end{pmatrix} + c_2^0 \begin{pmatrix} -\frac{1}{4}h^2 + \cdots \\ -\frac{3}{8}h^2 + \cdots \end{pmatrix}, \tag{5.5}$$

where c_1^0, c_2^0 are real constants, $c_1^0 > 0$.

(iii) *Near $h \to \infty$, we have*

$$(5.6)\qquad \begin{pmatrix} \widetilde{J}_0 \\ \widetilde{J}_1 \end{pmatrix} = c_1^{\infty}(-h)^{7/6}\begin{pmatrix} 1 - \frac{35}{108}h^{-1} + \cdots \\ \frac{7}{18}h^{-1} + \cdots \end{pmatrix} + c_2^{\infty}(-h)^{5/6}\begin{pmatrix} 2 + o(h^{-1}) \\ 1 + o(h^{-1}) \end{pmatrix},$$

where $c_1^{\infty}, c_2^{\infty}$ are real constants.

Proof: Since the value $h = -1/3$ corresponds to the center, we know that $\widetilde{J}_i(h)$ is analytic at $h = -1/3$; see [R]. By the integral mean-value theorem, one gets $\widetilde{J}_1/\widetilde{J}_0 \to 1$ as $h \to -1/3$. From (3.8), the vector $(\widetilde{J}_0, \widetilde{J}_1)$ satisfies the following equation:

$$(5.7)\qquad 6h(3h+1)\begin{pmatrix} J_0' \\ J_1' \end{pmatrix} = \begin{pmatrix} 21h & -3h \\ -7 & 3(5h+2) \end{pmatrix}\begin{pmatrix} J_0 \\ J_1 \end{pmatrix}.$$

Using analytic theory of ordinary differential equations [Ga, H], we get (5.5) and (5.6). Noting $J_i(0) > 0$, we have $c_1^0 > 0$. Since $\widetilde{J}_i(h)$ is real analytic at $h = -1/3$ and system (5.7) has no other singular point in $(-\infty, 0)$, we conclude that $J_i(h)$ is a real analytic function in $(-\infty, 0)$, which implies c_i^0 and c_i^{∞} are real constants, $i = 0, 1$.

Since (5.7) is a linear system with simple singular point, its solutions, including the vector (J_0, J_1), are (single-valued or multiply-valued) analytic functions on complex domain $\mathbf{C}\backslash\{h = 0, \infty\}$. To get the single-valued function on $\mathbf{C}$, define

$$\mathcal{D} = \mathbf{C}\backslash\{h | h \geq 0\}.$$

By the above discussion, we have

LEMMA 5.2: *$\widetilde{J}_i(h), i = 0, 1$, is a single-valued analytic function on $\mathcal{D}$.*

LEMMA 5.3: *In the expansion (5.6), $c_1^{\infty}c_2^{\infty} > 0$.*

Proof: In the proof of Lemma 5.1, we have known that $\widetilde{J}_0(h)$ and $\widetilde{J}_1(h)$ are real analytic functions in the interval $(-\infty, 0)$. Therefore, the ratio $u(h) = \widetilde{J}_1(h)/\widetilde{J}_0(h)$, $h \in (-\infty, 0)$, satisfies the Riccati equation (3.11), which implies that the curve $u(h)$ in the hv-plane is a trajectory of the system

$$(5.8)\qquad \begin{cases} \dot{h} = 6h(3h+1), \\ \dot{v} = 3hv^2 - 6(h-1)v - 7. \end{cases}$$

The system (5.8) has three critical points in the finite plane: an unstable node at $(0, 7/6)$, a saddle at $(-1/3, 1)$ and a stable node at $(-1/3, 7)$. The vertical zero isoclines $h = -1/3$ and $h = 0$ are invariant lines of (5.8). The zero isocline $v^{\pm}(h)$, on which the vector field is horizontal, is defined by the algebraic curve

$$\mathcal{K}(h, v) = 3hv^2 - 6(h-1)v - 7 = 0, \tag{5.9}$$

where

$$v^{\pm}(h) = \frac{3(h-1) \pm \sqrt{3(3h^2+h+3)}}{3h}, \tag{5.10}$$

which has the following properties:

(1) $v^+(-1/3) = 1, v^-(-1/3) = 7$, $\lim_{h\to+\infty} v^+(h) = 2$, $\lim_{h\to-\infty} v^+(h) = 0$, $\lim_{h\to+\infty} v^-(h) = 0$, $\lim_{h\to-\infty} v^-(h) = 2$;

(2) $v^{\pm}(h)$ has the following expansions near $h = 0$:

$$v^+(h) = \frac{7}{6} + o(1), \quad v^-(h) = -\frac{2}{h} + \frac{5}{6} + o(1), \tag{5.11}$$

which yields $v^+(0) = 7/6$, $\lim_{h\to 0^-} v^-(h) = +\infty$, $\lim_{h\to 0^+} v^+(h) = -\infty$;

(3) if $h \in (-\infty, 0)$, then $v^+(h) < v^-(h), dv^{\pm}(h)/dh > 0$; if $h \in (0, +\infty)$, then $v^+(h) > v^-(h), dv^{\pm}(h)/dh > 0$.

The properties (1) and (2) are obtained by direct computation. To prove (3), assume that there exists $h = \tilde{h}$ such that $dv^{\pm}(\tilde{h})/dh = 0$. Differentiating (5.9) with respect to h, we have $v^{\pm}(\tilde{h}) = 2$ or $v^{\pm}(\tilde{h}) = 0$. However, $\mathcal{K}(\tilde{h}, 2) = 5 > 0$ and $\mathcal{K}(\tilde{h}, 0) = -7 < 0$, which yields contradictions. Hence $dv^{\pm}(h)/dh \neq 0$. Property (3) follows from (1), (2) and (5.10).

Taking the Poincaré transformations

$$h = \frac{1}{\tilde{h}}, v = \frac{\tilde{v}}{\tilde{h}}, \quad dt = \tilde{h}^2 d\tau \quad \text{and} \quad h = \frac{v^*}{h^*}, \quad v = \frac{1}{h^*}, dt = {h^*}^2 d\tau,$$

system (5.8) changes to the form

$$\begin{cases} \dot{\tilde{h}} = -6\tilde{h}^2(\tilde{h}+3), \\ \dot{\tilde{v}} = 3\tilde{v}^2 - 24\tilde{h}\tilde{v} - 7\tilde{h}^3, \end{cases}$$

and

$$\begin{cases} \dot{h^*} = h^*(-3v^* + 6h^*(v^* - h^*) + 7{h^*}^3), \\ \dot{v^*} = v^*(-3v^* + 24v^*h^* + 7{h^*}^3), \end{cases}$$

respectively. Therefore, system (5.8) has two critical points $(\tilde{h}, \tilde{v}) = (0, 0)$ and $(h^*, v^*) = (0, 0)$ at infinity. By Lemma 5.1, $u(-1/3) = 1$, which means the curve $u(h)$ tends to the saddle $(-1/3, 1)$ as $h \to -1/3$. Since the trajectory of (5.8) crosses the zero isocline $v^+(h)$, $h \in (-\infty, -1/3)$, from the left hand to the right hand and $\dot{v}|_{v=2} = 5 > 0$, the curve $u(h)$ must stay in the region $\{(h,v)|0 < v^+(h) < v < 2 < v^-(h)\}$, which implies $u'(h) > 0, h \in (-\infty, -1/3)$. As there are only two critical points $(\widetilde{h}, \widetilde{u}) = (0,0)$ and $(h^*, u^*) = (0,0)$ at infinity, we conclude that $u(h)$ is the trajectory of system (5.8) starting from $(\widetilde{h}, \widetilde{u}) = (0,0)$ to the saddle $(-1/3, 1)$; see Figure 1. Hence $\lim_{h\to-\infty} u(h) = 0$. Noting $u(h) > v^+(h) > 0$, one gets $u(h) > 0$.

If $c_1^\infty c_2^\infty = 0$, then either $\lim_{h\to-\infty} u(h) = 1/2$ or $u(h) = (7/18)h^{-1} + \cdots < 0$ as $h \to -\infty$, which yields a contradiction. Therefore $c_1^\infty c_2^\infty \neq 0$. Using (5.6) again, we obtain

$$u(h) = \frac{\widetilde{J}_1}{\widetilde{J}_0} = \frac{c_2^\infty}{c_1^\infty}(-h)^{-1/3} + \cdots \tag{5.12}$$

as $h \to -\infty$. Since $u(h) > 0$ for $h \in (-\infty, -1/3)$, we have $c_1^\infty c_2^\infty > 0$. The lemma is proved.

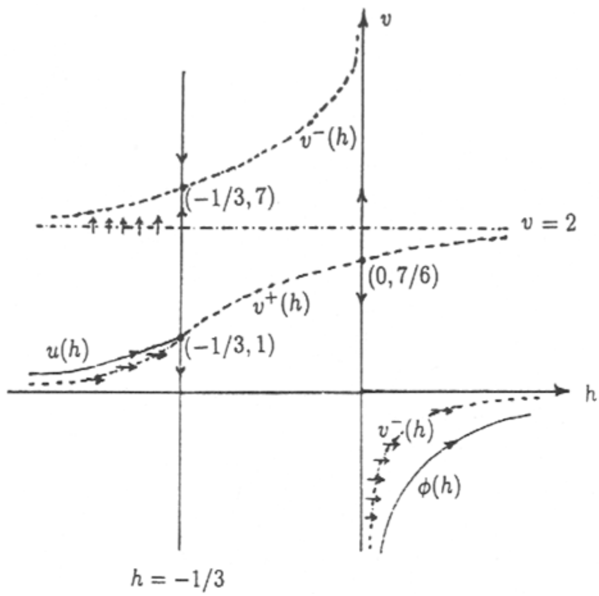

Figure 1

Lemma 5.4: *Suppose that S_+ (resp. S_-) is the upper (resp. the lower) side of the open cut $\{h|h \in (0, +\infty)\}$. Then for $h \in S_\pm$,*

(i) $Im(\widetilde{J}_0(h)/\widetilde{J}_1(h)) \neq 0$,

(ii) $Im\widetilde{J}_1(h) \neq 0$.

Proof: We only prove (i) and (ii) for $h \in S_+$.

(i) Suppose that there exists $h = h_1^*$ such that $\widetilde{J}_1(h_1^*) = 0$; then $Im\widetilde{J}_1(h_1^*) = Re\widetilde{J}_1(h_1^*) = 0$. Since $(\widetilde{J}_0(h), \widetilde{J}_1(h))$ is a solution of real analytic system (5.7), vectors $(Im\widetilde{J}_0(h), Im\widetilde{J}_1(h))$ and $(Re\widetilde{J}_0(h), Re\widetilde{J}_1(h))$ are two real analytic solutions of system (5.7), too. It follows from Liouville's formula that

$$W(h) = \begin{vmatrix} Re\widetilde{J}_0 & Im\widetilde{J}_0 \\ Re\widetilde{J}_1 & Im\widetilde{J}_1 \end{vmatrix} = W(h_1^*)e^{\int_{h_1^*}^{h} \frac{6h+1}{h(3h+1)}dh} \equiv 0. \tag{5.13}$$

Hence, in the region $\{h|\widetilde{J}_1(h) \neq 0, h \in S_+\}$, we have

$$Im\frac{\widetilde{J}_0(h)}{\widetilde{J}_1(h)} = -\frac{W(h)}{|\widetilde{J}_1(h)|^2} \equiv 0. \tag{5.14}$$

It follows from (5.6) and Lemma 5.3 that $\widetilde{J}_0(h)/\widetilde{J}_1(h) \sim (c_1^\infty/c_2^\infty)(-h)^{1/3}$ as $h \to +\infty$. This implies that $Im(\widetilde{J}_0(h)/\widetilde{J}_1(h)) \neq 0$ as $h \to +\infty$, which contradicts (5.14). Hence, $\widetilde{J}_1(h) \neq 0$ for $h \in S_+$.

Based on $\widetilde{J}_1(h) \neq 0, h \in S_+$, we can define the ratio $\widetilde{J}_0(h)/\widetilde{J}_1(h)$. Noting $Im(\widetilde{J}_0(h)/\widetilde{J}_1(h)) = -W(h)/|\widetilde{J}_1|^2$ and using the same arguments as above, we get $Im(\widetilde{J}_0(h)/\widetilde{J}_1(h)) \neq 0$ for $h \in S_+$.

(ii) It is obvious that $\phi(h) = Im\widetilde{J}_1(h)/Im\widetilde{J}_0(h), h \in S_+$, is a trajectory of system (5.8). Using (5.5) and (5.6), we have

$$\phi(h) = -\frac{4}{h} + o(h^{-1}) < v^-(h) = -\frac{2}{h} + o(h^{-1})$$

as $h \to 0^+$ and

$$\phi(h) = -(c_2^\infty/c_1^\infty)h^{-1/3} + o(h^{-1/3}) \to 0$$

as $h \to +\infty$, which implies that $\phi(h)$ must stay in the region

$$\{(h, v)|v < v^-(h) < 0\};$$

see Figure 1. Therefore, $\phi'(h) > 0$ and $\phi(h) \neq 0$. This yields

$$-\infty < \frac{Im\widetilde{J}_1(h)}{Im\widetilde{J}_0(h)} < 0, \quad h \in S_+. \tag{5.15}$$

It follows from (5.15) that if there exists $h = h_2^*$ such that $Im\widetilde{J}_1(h_2^*) = 0$, then $Im\widetilde{J}_0(h_2^*) = 0$, which implies $W(h) \equiv 0$; cf. (5.13). Using (5.14) again, we obtain $Im(\widetilde{J}_0(h)/\widetilde{J}_1(h)) \equiv 0, h \in (0, +\infty)$. On the other hand, the expansion (5.6) shows that $Im(\widetilde{J}_0(h)/\widetilde{J}_1(h)) \neq 0$ as $h \to +\infty$, which yields a contradiction. The conclusion (ii) follows.

LEMMA 5.5: *Suppose $h \neq -1/3, h \in \mathcal{D}$; then $\widetilde{J}_1(h) \neq 0$.*

Proof: Let d_∞ be a big enough constant and d_0 be a small enough constant. Denote by $\mathcal{D}_1$ the set obtained from $\mathcal{D} \cap \{|h| < d_\infty\}$ by removing a circle of radius d_0 around $h_0 = 0$; see Figure 2. Consider the increase in the arguments of $\widetilde{J}_1(h)$ along the boundary of $\mathcal{D}_1$ which has positive (counter clockwise) orientation. Lemma 5.2 shows that $\widetilde{J}_1(h)$ is single-valued analytic in the set $\mathcal{D}$. It follows from (5.5) that the change of argument of $\widetilde{J}_1(h)$, when h makes one turn along the circle $|h| = d_0$, is close to zero. The expansion (5.6) yields that along the circle $|h| = d_\infty$, the change in the argument of $\widetilde{J}_1(h)$ is close to $5\pi/3$. At the end, on the upper and the lower side of open cut $\{h | h \in (d_0, d_\infty)\}$, $Im\widetilde{J}_1(h) \neq 0$. Putting these data together yields that the increment in the argument of $\widetilde{J}_1(h)$ along the boundary of $\mathcal{D}_1$ is less than $5\pi/3 + 2\pi + \epsilon$, $|\epsilon| \ll 1$, as $d_0 \to 0, d_\infty \to +\infty$. Using the argument principle, we obtain that $\widetilde{J}_1(h)$ has at most one zero in $\mathcal{D}_1$. The same is true, of course, for $\mathcal{D}$. Since $\widetilde{J}_1(-1/3) = 0$, the result of this lemma follows.

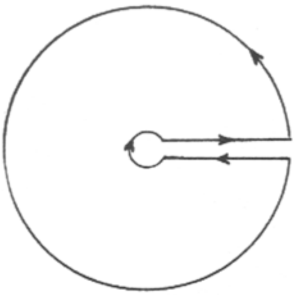

Figure 2

LEMMA 5.6: $\#G(h) \leq 4[(n-1)/2] - 1, n \geq 4, h \in \Sigma = (-1/3, 0)$.

Proof: Since $J_1(h) \neq 0$ in Σ, the number of zeros of $G(h)$ is equal to the number of zeros of $G(h)/J_1(h)$. Let $\widetilde{G}_1(h)$ be the analytic continuation of $G(h)/J_1(h)$ from Σ to the complex domain $\mathbf{C}$, namely

$$\widetilde{G}_1(h) = \alpha_2(h)\frac{\widetilde{J}_0(h)}{\widetilde{J}_1(h)} + \beta_2(h).$$

By Lemma 5.1, $(\widetilde{J}_0(h)/\widetilde{J}_1(h))|_{h=-1/3} = 1$. Since $\widetilde{J}_1(h) \neq 0$ in $\mathcal{D}\backslash\{-1/3\}$, we conclude that $\widetilde{G}_1(h)$ is single-valued analytic in $\mathcal{D}$.

To estimate the number of zeros in $\mathcal{D}$, we should evaluate the increment in the argument of the function $\widetilde{G}_1(h)$ along the boundary of $\mathcal{D}_1$. In what follows we split the proof into two cases.

CASE 1: Assume that $\alpha_2(h)$ and $\beta_2(h)$ have no common factor.

Since $Im(\widetilde{J}_0(h)/\widetilde{J}_1(h)) \neq 0$ for $h \in S_+$ (resp. S_-), we know that $Im\widetilde{G}_1(h)$ has at most $2[(n-1)/2]-1$ zeros in S_+ (resp. S_-). The expansion (5.6) shows that $\widetilde{G}_1(h) \sim h^l$ as $h \to \infty$, where $l \leq \max\{\deg \alpha_2(h) + 1/3, \deg \beta_2(h)\} \leq 2[(n-1)/2] - 2/3$. This implies that along the circle $|h| = d_\infty$, the change in the argument of $\widetilde{G}_1(h)$ is close to $2\pi(2[(n-1)/2]-2/3)$. Noticing that the circle $|h| = h_0$ has negative orientation, it follows that along the circle $|h| = h_0$ the increment in the argument of $\widetilde{G}_1(h)$ increases by no more than zero. By the same arguments as in the proof of Lemma 5.5, one gets that the increment in the argument of $\widetilde{G}_1(h)$ along the boundary of $\mathcal{D}_1$ is close to

$$2\pi\Big(2\Big[\frac{n-1}{2}\Big] - \frac{2}{3} + 2\Big[\frac{n-1}{2}\Big] - 1 + 1\Big) = 2\pi\Big(4\Big[\frac{n-1}{2}\Big] - \frac{2}{3}\Big)$$

and hence $\widetilde{G}_1(h)$ has at most $4[(n-1)/2]-1$ zeros in $\mathcal{D}_1$, which implies that $\#G(h) \leq \#\widetilde{G}_1(h) \leq 4[(n-1)/2]-1, h \in \Sigma = (-1/3, 0)$.

CASE 2: Assume that $\alpha_2(h)$ and $\beta_2(h)$ have common factor $G_2(h)$, $\deg G_2(h) \leq m$.

Denote $G(h) = G_2^*(h)G_2(h)$. Using the same arguments as in Case 1, we have $\#G_2(h) \leq 4[(n-1)/2]-2m-1$, which yields $\#G(h) \leq 4[(n-1)/2]-m-1, h \in \Sigma$.

Proof of Theorem 1 for Q_4^+: By (5.4) and Lemma 5.6 one gets

$$\#I(h) = \#J(h) \leq 4\Big[\frac{n-1}{2}\Big] - 1 + \Big[\frac{n-3}{2}\Big] + 1 = 5\Big[\frac{n-1}{2}\Big] - 1,$$

where $n \geq 7$. Using the same arguments, we obtain $\#I(h) \leq 4$ for $1 \leq n \leq 6$.

References

[A] V. I. Arnold, *Loss of stability of self-oscillation close to resonance and versal deformations of equivariant vector fields*, Functional Analysis and its Applications **11** (1977), 1–10.

[G1] L. Gavrilov, *Nonoscillation of elliptic integrals related to cubic polynomials with symmetry of order three*, The Bulletin of the London Mathematical Society **30** (1998), 267–273.

[G2] L. Gavrilov, *Petrov modules and zeros of Abelian integrals*, Bulletin des Sciences Mathématiques **122** (1998), 571–584.

[Ga] W. Gao, *Analytic Theory of Ordinary Differential Equations*, (in Chinese), to appear.

[H] P. Hartman, *Ordinary Differential Equations*, 2nd edn., Birkhäuser, Basel, 1982.

[HI] E. Horozov and I. D. Iliev, *Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonians*, Nonlinearity **11** (1998), 1521–1537.

[I1] I. D. Iliev, *Inhomogeneous Fuchs equations and the limit cycles in a class of near-integrable quadratic systems*, Proceedings of the Royal Society of Edinburgh **127A** (1997), 1207–1217.

[I2] I. D. Iliev, *Perturbations of quadratic centers*, Bulletin des Sciences Mathématiques **122** (1998), 107–161.

[I3] I. D. Iliev, *On the limit cycles obtainable from polynomial perturbations of the Bogdanov–Takens Hamiltonian*, Israel Journal of Mathematics **115** (2000), 269–284.

[IyY1] Yu. S. Il'yashenko and S. Yakovenko, *Double exponential estimate for the number of zeros of complete Abelian integrals*, Inventiones Mathematicae **121** (1995), 613–650.

[IyY2] Yu. S. Il'yashenko and S. Yakovenko, *Counting real zeros of analytic functions satisfying linear ordinary differential equations*, Journal of Differential Equations **126** (1996), 87–105.

[NY] D. Novikov and S. Yakovenko, *Simple exponential estimate for the number of real zeros of complete Abelian integrals*, Annales de l'Institut Fourier **45** (1995), 897–927.

[P1] G. S. Petrov, *The Chebyschev property of elliptic integrals*, Functional Analysis and its Applications **22** (1988), 72–73.

[P2] G. S. Petrov, *Complex zeros of an elliptic integral*, Functional Analysis and its Applications **23** (1989), 88–89.

[R] R. Roussarie, *Bifurcation of planar vector fields and Hilbert sixteen problem*, Progress in Mathematics 164, Birkhäuser Verlag, Basel, 1998.

[RZh] C. Rousseau and H. Zoladek, *Zeros of complete elliptic integrals for 1:2 resonance*, Journal of Differential Equations **94** (1991), 42–54.

[Y] S. Yakovenko, *Complete Abelian integrals as rational envelopes*, Nonlinearity **7** (1994), 1237–1250.

[Zh] H. Zoladek, *Quadratic systems with centers and their perturbations*, Journal of Differential Equations **109** (1994), 223–273.

[ZLL] Y. Zhao, Z. Liang and G. Lu, *The cyclicity of the period annulus of the quadratic Hamiltonian system with non-Morsean point*, Journal of Differential Equations **162** (2000), 199–223.

[ZLLZ] Y. Zhao, W. Li, C. Li and Z. Zhang, *Linear estimate of the number of zeros of Abelian integrals for quadratic centers having almost all their orbits formed by cubics*, Science in China, Series A **45** (2002), 964–974.

[ZZ] Y. Zhao and Z. Zhang, *Linear estimate of the number of zeros of Abelian integrals for a kind of quartic Hamiltonian*, Journal of Differential Equations **155** (1999), 73–88.